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BEHAVIOR OF A PULSATING RIGID BODY IN A VISCOUS FLUID IN THE PRESENCE OF GRAVITY

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The paper considers the motion of a rigid sphere due to specified pulsations of the sphere and gravity in a viscous fluid which is at rest away from the sphere.

Experimental and theoretical studies [1–8] demonstrated that vibrations of a fluid with a rigid body can lead to qualitative changes in the behavior of the body.

The problem of the motion of a pulsating rigid sphere in an unbounded viscous incompressible fluid which vibrates at infinity in the absence of gravity was formulated and solved in [7]. In the present paper, we consider the motion of a pulsating rigid sphere under gravity in an unbounded viscous incompressible fluid which is at rest at infinity. It is found that behavior of pulsating spheres can be qualitatively different from the behavior of spheres of constant volume, and a pulsating sphere can behave paradoxically.

1. A compressible rigid sphere is placed in an unbounded viscous incompressible fluid. The fluid is at rest at infinity relative to the inertial Cartesian coordinate system X, Y, Z. The sphere radius change periodically in a prescribed manner with period T in time t. The distribution of the sphere material is symmetric about its center (the center of inertia coincides with the center of the sphere). There is a constant field of gravity; the acceleration of gravity is $\boldsymbol{g} = (0, 0, -g)$, where $g > 0$. The fluid flow does not depend on the initial conditions. The position of the sphere is characterized by the radius-vector $\boldsymbol{S}$ of its center. The problem is to find how $\boldsymbol{S}$ depends on t.

The present formulation of the problem corresponds to the following: a closed container is filled with a fluid and contains a body, and deformations of the container cause the body to pulsate (the body volume changes periodically with time) (see [7]). We note, however, that the forces acting inside the body can also cause its pulsations. In this case, the body moves in the fluid which experiences vibration effects exerted by the body itself.

We consider the fluid flow and the motion of the sphere relative to the Cartesian coordinate system $X_1 = X - S_X$, $X_2 = Y - S_Y$, $X_3 = Z - S_Z$ (S_X, S_Y, and S_Z are, respectively, the X, Y, and Z components of the vector $\boldsymbol{S}$).

We assume that $\tau = t/T$, $A = A_0(1 + æa)$ is the radius of the sphere [A_0 ($A_0 > 0$) is a constant, $æ$ ($æ < 1$) is the maximum value of $|A - A_0|/A_0$, and $a = \text{Real} \sum\limits_{m=1}^{\infty} a_m \mathrm{e}^{2m\pi i\tau}$ (a_m are constants)], $x_1 = X_1/A_0$, $x_2 = X_2/A_0$, $x_3 = X_3/A_0$, $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, ρ is the density of the fluid, m is the mass of the sphere, $\mu = 3m/(4\pi A_0^3 \rho)$, (s) is the surface of the sphere [the equation of (s) is $r = 1 + æa$], $\boldsymbol{n}$ is the unit outward normal to (s), $\boldsymbol{\alpha} = \alpha \boldsymbol{k} = -T^2 \boldsymbol{g}/A_0$ [$\boldsymbol{k} = (0, 0, 1)$], $\boldsymbol{V}$ is the fluid velocity, $\boldsymbol{v} = T\boldsymbol{V}/A_0$, P is the fluid pressure, $p = T^2 P/(\rho A_0^2)$, $\boldsymbol{W} = d\boldsymbol{S}/dt$, $\boldsymbol{w} = T\boldsymbol{W}/A_0$, ν is the kinematic viscosity of the fluid, $\text{Re} = A_0^2/(\nu T)$ is the Reynolds number, $\mathcal{P}$ is the stress tensor for the fluid, $\wp = T^2 \mathcal{P}/(\rho A_0^2)$, $\boldsymbol{F}$ is the force exerted by the fluid on the body, and $\boldsymbol{f} = T^2 \boldsymbol{F}/(\rho A_0^4) = \iint\limits_{(s)} \wp \cdot \boldsymbol{n}\, ds$.

We write the equation of motion for the center of inertia of the sphere, the Navier–Stokes and continuity equations, and the conditions that must be satisfied at (s) and for $r \to \infty$:

$$\boldsymbol{f} - \frac{4\pi}{3}\,\mu\Big(\frac{d\boldsymbol{w}}{d\tau} + \boldsymbol{\alpha}\Big) = 0,$$

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$$\frac{\partial \boldsymbol{v}}{\partial \tau} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla p + \frac{1}{\mathrm{Re}}\,\Delta \boldsymbol{v} - \frac{d\boldsymbol{w}}{d\tau} - \boldsymbol{\alpha}, \quad \nabla\cdot\boldsymbol{v} = 0; \tag{1.1}$$

$$\boldsymbol{v} = æ\,\frac{da}{d\tau}\,\boldsymbol{n} \quad \text{on} \quad (s), \qquad \boldsymbol{v} \sim -\boldsymbol{w} \quad \text{as} \quad r\to\infty. \tag{1.2}$$

2. We consider problem (1.1), (1.2) for small (compared to unity) values of Re.

We assume that

$$\boldsymbol{v} \sim \boldsymbol{v}_{(0)} + \mathrm{Re}\,\boldsymbol{v}_{(1)}, \quad p \sim \frac{1}{\mathrm{Re}}\,p_{(0)} + p_{(1)}, \quad \boldsymbol{w} \sim \boldsymbol{w}_{(0)} + \mathrm{Re}\,\boldsymbol{w}_{(1)} \tag{2.1}$$

as $\mathrm{Re}\to 0$.

According to (1.1), (1.2), and (2.1), as the Lth approximation ($L = 0$, 1), we have

$$\boldsymbol{f}_{(L)} - L\,\frac{4\pi}{3}\,\mu\Big(\frac{d\boldsymbol{w}_{(0)}}{d\tau} + \boldsymbol{\alpha}\Big) = 0,$$

$$\nabla p_{(L)} - \Delta\boldsymbol{v}_{(L)} + L\Big[\frac{\partial \boldsymbol{v}_{(0)}}{\partial\tau} + (\boldsymbol{v}_{(0)}\cdot\nabla)\boldsymbol{v}_{(0)} + \frac{d\boldsymbol{w}_{(0)}}{d\tau} + \boldsymbol{\alpha}\Big] = 0, \tag{2.2}$$

$$\nabla\cdot\boldsymbol{v}_{(L)} = 0;$$

$$\boldsymbol{v}_{(L)} = (1-L)æ\,\frac{da}{d\tau}\,\boldsymbol{n} \quad \text{for} \quad r = 1 + æa, \quad \boldsymbol{v}_{(L)} \sim -\boldsymbol{w}_{(L)} \quad \text{as} \quad r\to\infty. \tag{2.3}$$

Here

$$\boldsymbol{f}_{(L)} = \iint\limits_{(s)} \wp_{(L)}\cdot\boldsymbol{n}\,ds, \tag{2.4}$$

where $\wp_{(L)}$ is $\wp$ for $\boldsymbol{v} = \boldsymbol{v}_{(L)}$, $p = p_{(L)}$, and $\mathrm{Re} = 1$ [$\wp \sim (1/\mathrm{Re})\wp_{(0)} + \wp_{(1)}$ and $\boldsymbol{f} \sim (1/\mathrm{Re})\boldsymbol{f}_{(0)} + \boldsymbol{f}_{(1)}$ as $\mathrm{Re}\to 0$)].

Let $L = 0$. To a zeroth approximation, displacement of the sphere with nonzero velocity relative to the fluid at infinity cannot occur because of gravity; the center of inertia of the sphere is at rest relative to the coordinate system X, Y, Z, and the fluid flow is symmetric about the origin of the coordinates x_1, x_2, x_3. Problem (2.2)–(2.4) has the solution

$$\boldsymbol{v}_{(0)} = æ\,\frac{da}{d\tau}\,(1+æa)^2 r^{-3}\boldsymbol{r}, \qquad p_{(0)} = c_{(0)};$$

$$\boldsymbol{f}_{(0)} = 0; \tag{2.5}$$

$$\boldsymbol{w}_{(0)} = 0. \tag{2.6}$$

Here $\boldsymbol{r} = (x_1, x_2, x_3)$ and $c_{(0)}$ is a function of τ.

Let $L = 1$. Problem (2.2)–(2.4) has the solution

$$v_{(1)r} = w_{(1)}\{-1 + (1/2)(1+æa)[3 - (1+æa)^2 r^{-2}]r^{-1}\}\cos\theta,$$

$$v_{(1)\theta} = w_{(1)}\{1 - (1/4)(1+æa)[3 + (1+æa)^2 r^{-2}]r^{-1}\}\sin\theta, \qquad v_{(1)\varphi} = 0,$$

$$p_{(1)} = \Big[-\alpha r + \frac{3}{2}\,w_{(1)}(1+æa)r^{-2}\Big]\cos\theta + æ\,\frac{d[(da/d\tau)(1+æa)^2]}{d\tau}\,r^{-1} - \frac{1}{2}\,æ^2\Big(\frac{da}{d\tau}\Big)^2(1+æa)^4 r^{-4} + c_{(1)};$$

$$\boldsymbol{f}_{(1)} = (4\pi/3)(1+æa)^3\boldsymbol{\alpha} - 6\pi(1+æa)\boldsymbol{w}_{(1)}; \tag{2.7}$$

$$\boldsymbol{w}_{(1)} = (2/9)[(1+æa)^2 - \mu(1+æa)^{-1}]\boldsymbol{\alpha}. \tag{2.8}$$

Here θ is the angle between the vectors $(0,0,1)$ and (x_1,x_2,x_3), φ is the angle between the vectors $(1,0,0)$ and $(x_1,x_2,0)$ (r, θ, and φ are spherical coordinates), $v_{(1)r}$, $v_{(1)\theta}$, and $v_{(1)\varphi}$ are the r, θ, and φ components of the vector $\boldsymbol{v}_{(1)}$, respectively, and $c_{(1)}$ is a function of τ.

3. Using (2.6) and (2.8), we obtain

$$\boldsymbol{W} = (2/9)[\mu(1+æa)^{-1} - (1+æa)^2]\,A_0^2\boldsymbol{g}/\nu. \tag{3.1}$$

At $\ae = 0$ for a sphere of constant volume, (3.1) becomes

$$\boldsymbol{W} = W_0 \boldsymbol{k}, \tag{3.2}$$

where $W_0 = (2/9)(1-\mu)A_0^2 g/\nu$. We note that (3.2) coincides with the Stokes formula for the velocity at which the sphere of constant volume moves under gravity in a viscous fluid at small Reynolds numbers [9, 10].

From (3.1), it follows that

$$\boldsymbol{S} = \bar{W} t \boldsymbol{k} + \tilde{\boldsymbol{S}}, \tag{3.3}$$

where

$$\bar{W} = (2/9)[\langle(1+\ae a)^2\rangle - \mu\langle(1+\ae a)^{-1}\rangle]A_0^2 g/\nu \quad \Big(\langle\cdots\rangle = \int\limits_{\tau}^{\tau+1} \cdots d\tau\Big),$$

$$\tilde{\boldsymbol{S}} = \boldsymbol{S}_0 + \mathrm{Real} \sum_{m=1}^{\infty} S_m \mathrm{e}^{2m\pi i\tau} \boldsymbol{k}$$

($\boldsymbol{S}_0$ and S_m are constants). Relation (3.3) approximately defines the dependence of $\boldsymbol{S}$ on t.

According to (3.3), the sphere moves along the Z axis, and its motion consists of vibrations and displacement with constant velocity:

$$\bar{\boldsymbol{W}} = \bar{W} \boldsymbol{k}. \tag{3.4}$$

Using (3.2) and (3.4), we compare the behavior of the pulsating sphere with the behavior of the sphere of constant volume.

A. Let $\mu < 1$. Then, $W_0 > 0$ and the sphere of constant volume floats; the pulsating sphere can float faster than the sphere of constant volume (at $\bar{W} > W_0$), float more slowly than the sphere of constant volume (at $0 < \bar{W} < W_0$), neither float nor sink (at $\bar{W} = 0$), or sink (at $\bar{W} < 0$).

B. Let $\mu = 1$. Then, $W_0 = 0$ and the sphere of constant volume is at rest; the pulsating sphere can float (at $\bar{W} > 0$) or sink (at $\bar{W} < 0$).

C. Let $\mu > 1$. Then, $W_0 < 0$ and the sphere of constant volume sinks; the pulsating sphere can sink faster than the sphere of constant volume (at $\bar{W} < W_0$), sink more slowly than it (at $W_0 < \bar{W} < 0$), neither sink nor float (at $\bar{W} = 0$), or float (at $\bar{W} > 0$).

D. Let $\mu \lesseqqgtr 1$. Then, $W_0 \gtreqqless 0$ and the pulsating sphere can move in the same manner as the sphere of constant volume (at $\bar{W} = W_0$).

4. We consider the force exertion by the fluid on the sphere.

In accordance with (2.5) and (2.7), we have

$$\boldsymbol{F} = \boldsymbol{F}_{\mathrm{A}} + \boldsymbol{F}_{\mathrm{C}}, \tag{4.1}$$

where $\boldsymbol{F}_{\mathrm{A}} = F_{\mathrm{A}}\boldsymbol{k} = -(4\pi/3)A^3\rho\boldsymbol{g}$ is the buoyancy Archimedean force and $\boldsymbol{F}_{\mathrm{C}} = -6\pi\rho\nu A\boldsymbol{W}$ is the Stokes drag force.

From (3.1) and (4.1), it follows that

$$\langle\boldsymbol{F}\rangle = \langle\boldsymbol{F}_{\mathrm{A}}\rangle - 6\pi\rho\nu A_0\bar{\boldsymbol{W}} + \boldsymbol{F}_{\mathrm{vibr}}, \tag{4.2}$$

where

$$\boldsymbol{F}_{\mathrm{vibr}} = -6\pi\rho\nu A_0\ae\langle a\tilde{\boldsymbol{W}}\rangle = (4\pi/3)\ae[2\ae\langle a^2\rangle + \ae^2\langle a^3\rangle - \mu\langle a(1+\ae a)^{-1}\rangle]A_0^3\rho\boldsymbol{g} \quad (\tilde{\boldsymbol{W}} = \boldsymbol{W} - \bar{\boldsymbol{W}}). \tag{4.3}$$

We write $\langle\boldsymbol{F}_{\mathrm{A}}\rangle$ in the form

$$\langle\boldsymbol{F}_{\mathrm{A}}\rangle = \boldsymbol{F}_{\mathrm{A}0} - (4\pi/3)\ae^2(3\langle a^2\rangle + \ae\langle a^3\rangle)A_0^3\rho\boldsymbol{g}, \tag{4.4}$$

where $\boldsymbol{F}_{\mathrm{A}0} = F_{\mathrm{A}0}\boldsymbol{k} = -(4\pi/3)A_0^3\rho\boldsymbol{g}$. According to (4.4), $\langle F_{\mathrm{A}}\rangle > F_{\mathrm{A}0}$. In particular, this relation indicates that for $\mu < 1$, if the sphere of constant volume floats, all the more, the pulsating sphere should float. However, as noted above, at $\mu < 1$, the pulsating sphere can, for instance, sink. Thus, the pulsating sphere in the fluid in the presence of the body force can behave unusually, paradoxically. The reason for such behavior is as follows. Because of periodical changes in volume and the presence of gravity, the sphere vibrates along the vertical axis relative to the fluid at infinity, and the simultaneous occurrence of such vibrations and changes in the volume of the sphere results in the force $\boldsymbol{F}_{\rm vibr}$ exerted by the fluid on the sphere [se, (4.1)–(4.3)]. In accordance with (4.3), $\boldsymbol{F}_{\rm vibr}$ is independent of t, the direction of $\boldsymbol{F}_{\rm vibr}$ coincides with the direction of $\boldsymbol{g}$ and is opposite to the direction of $\langle \boldsymbol{F}_{\rm A}\rangle$. The action of the force $\boldsymbol{F}_{\rm vibr}$ leads to the effect of "the sphere of constant volume floats–the pulsating sphere sinks" and other types of behavior of the pulsating sphere.

5. Let the values of æ be small in comparison to unity.

From (3.4), it follows that as $æ \to 0$,

$$\bar{\boldsymbol{W}} \sim (2/9)[(\mu-1)(1+æ^2\langle a^2\rangle) - \mu æ^3\langle a^3\rangle]A_0^2\boldsymbol{g}/\nu. \qquad (5.1)$$

The dependence of a on τ can be specified in such a manner that $\langle a^3\rangle$ is positive, negative, or zero. For instance, $\langle a^3\rangle \gtrless 0$ if $a = (\lambda/5)(4\cos 2\pi\tau + \cos 4\pi\tau)$ $(\lambda = \pm 1)$ and $\langle a^3\rangle = 0$ if $a = \lambda\cos 2\pi\tau$.

Using (3.2) and (5.1), one can see that all the types of behavior of the pulsating sphere noted above are possible. In particular, according to (3.2) and (5.1), the effect "the sphere of constant volume floats–the pulsating sphere sinks" occurs for $1 + æ^3\langle a^3\rangle < \mu < 1$.

6. The results obtained show that in the presence of body force, even implementation of body pulsations is enough to produce qualitative changes in its behavior.

As noted above (see, for example, [11, 12]), vibration effects on a fluid with inclusions can be used as a method for controlling the inclusions. The results presented here demonstrate a new possibility of controlling inclusions in a fluid by means of vibration.

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